

Dynamics of a Compact Operator

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Abstract

Let $T : X \rightarrow X$ be a compact linear (or more generally affine) operator from a Banach space into itself. For each $x \in X$, the sequence of iterates $T^n x, n = 0, 1, \dots$ and its averages $\frac{1}{k} \sum_{n=0}^{k-1} T^n x, n = 0, 1, \dots$ are either bounded or approach infinity.

Keywords: compact operator, dynamics of linear operator, average, iterate

Let X be a set and $f : X \rightarrow X$ a map from X into X . For an $x \in X$, the sequence of iterations $x, f(x), f^2(x), \dots, f^k(x), \dots$ can be considered as a trajectory of a dynamical system where time is the discrete nonnegative integers: starting with the initial (time $t = 0$) state x , the state at time $t = k$ is $f^k(x)$. Suppose now $X = \mathbb{C}^n$ and f is the transformation defined by an $n \times n$ complex matrix A . What can one say about the general behavior of trajectories of A ? More generally, we consider affine maps on X , i.e. maps of the form $Ax + c$, where A is linear and c is a constant vector, and we also allow X to be infinite dimensional. Moreover, we study the behavior of the sequence of averages:

$$\text{Ave}_k f(x) = \frac{1}{k}(x + f(x) + \dots + f^{k-1}(x)), k = 1, 2, \dots$$

Note that the method of averaging was used in [2] in approximating solutions of a system of linear equations. In case of linear operators, problems about the linear span of the iterates $T^k x, n = 0, 1, \dots$ can be found in [4].

Recall that a square matrix N is called *nilpotent* if $N^s = 0$ for some nonnegative integer s .

Throughout this paper, for nonnegative integers $k, j, k \geq j$, $C(k, j)$ denotes the binomial coefficient

$$\frac{k!}{j!(k-j)!}$$

By convention, $C(k, 0) = 1$ for $k \geq 0$ and $C(k, j) = 0$ if $k < j$.

If $k < j$, the sum $\sum_{i=j}^k u_i$ is considered as an empty sum and its value is 0.

Theorem 1 *Let $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an affine map defined by $Tx = Ax + c$, where A is an $n \times n$ complex matrix and c a constant vector in \mathbb{C}^n . Let $\|\cdot\|$ be a norm on \mathbb{C}^n . Then for any vector $x \in \mathbb{C}^n$, the sequence*

$$T^k x, k = 0, 1, 2, \dots$$

is either bounded or $\lim_k \|T^k x\| = \infty$.

Proof.

By Jordan canonical decomposition theorem, $\mathbb{C}^n = V_1 \oplus V_2 \oplus \dots \oplus V_m$ for some subspaces $V_i, i = 1, 2, \dots, m$ with the following properties: (a) each V_i is an invariant subspace of A , i.e. $Av \in V_i$ for all $v \in V_i$, and (b) there exists $\lambda_i \in \mathbb{C}$ and a nilpotent matrix N_i such that $Av = \lambda_i v + N_i v$ for all $v \in V_i$.

Let P_i be the algebraic projection of \mathbb{C}^n onto V_i associated with the decomposition $\mathbb{C}^n = V_1 \oplus V_2 \oplus \dots \oplus V_m$.

Define a new norm $|\cdot|$ on \mathbb{C}^n by

$$|v| = \|P_1 v\| + \|P_2 v\| + \dots + \|P_m v\|.$$

Let x be a vector in \mathbb{C}^n . Let

$$x_k = T^k x = A^k x + c + Ac + \dots + A^{k-1} c, k = 1, 2, \dots$$

For any vector x , we have the following equalities:

$$\begin{aligned} x &= P_1 x + \dots + P_m x \\ A^k x &= A^k P_1 x + \dots + A^k P_m x \\ A^k x &= P_1 A^k x + \dots + P_m A^k x \end{aligned}$$

The last equality follows from the commutative property of A and $P_i, i = 1, \dots, m$ since A is invariant in each V_j .

Fix an i and write $v = P_i x, d = P_i c, \lambda = \lambda_i, V = V_i, N = N_i$. Let s be the smallest nonnegative integer such that $N^s v = 0$ and t the smallest nonnegative integer such that $N^t d = 0$. Then for $k > \max\{s, t\}$ and $s, t \geq 1$ (if $s = 0$ or $t = 0$, the corresponding sum below is defined as 0), one has

$$P_i x_k = (\lambda I + N)^k v + d + (\lambda I + N)d + \dots + (\lambda I + N)^{k-1} d \quad (1)$$

$$= \sum_{j=0}^{s-1} C(k, j) \lambda^{k-j} N^j v + \sum_{j=0}^{t-1} S(j, k) N^j d \quad (2)$$

where

$$S(j, k) = C(j, j) + C(j+1, j)\lambda + \dots + C(k-1, j)\lambda^{k-1-j}.$$

Using the identity $C(j, i+1) + C(j, i) = C(j+1, i+1)$, we have for $\lambda = 1$, $S(j, k) = C(k, j+1)$, and for $\lambda \neq 1$, $S(j, k)$ is given recursively by

$$S(0, k) = \frac{1 - \lambda^k}{1 - \lambda}$$

and (by subtracting $\lambda S(j, k)$ from $S(j, k)$)

$$(1 - \lambda)S(j, k) = S(j-1, k) - \lambda^{k-j}C(k, j), j = 1, 2, \dots, t-1, (\text{ for } t \geq 2),$$

from which we get an alternate formula for $S(j, k)$:

$$S(j, k) = \frac{1 - \lambda^k}{(1 - \lambda)^{j+1}} - \sum_{i=0}^{j-1} \frac{C(k, i+1)\lambda^{k-i-1}}{(1 - \lambda)^{j-i}}, j = 1, \dots, t-1 \quad (3)$$

Note that (3) is also valid for $j = 0$ for $\sum_{i=0}^{-1}$ is an empty sum. We shall show that for $\lambda \neq 1$,

$$P_i x_k = B + \sum_{j=0}^{w-1} \lambda^{k-j} C(k, j) A_j \quad (4)$$

where $w = \max\{s, t\}$, and $A_j, B, j = 0, \dots, w-1$ are constant vectors independent of k ; and for $\lambda = 1$,

$$P_i x_k = v + \sum_{j=1}^l C(k, j) B_j \quad (5)$$

where $l = \max\{s-1, t\}$, and $B_j, j = 1, \dots, l$ are constant vectors independent of k . More precisely, for $k > \max\{s, t\}$,

$$A_j = N^j v - \sum_{i=j}^{t-1} \frac{1}{(1 - \lambda)^{i-j+1}} N^i d, j = 0, \dots, \min\{s, t\} - 1,$$

$$A_j = \epsilon(s-t)N^j v - \epsilon(t-s) \sum_{i=j}^{t-1} \frac{1}{(1 - \lambda)^{i-j+1}} N^i d, j = \min\{s, t\}, \dots, w-1$$

and

$$B = \sum_{i=0}^{t-1} \frac{1}{(1 - \lambda)^{i+1}} N^i d;$$

and

$$B_j = N^j v + N^{j-1} d, j = 1, \dots, \min\{s-1, t\},$$

$$B_j = \epsilon(s-1-t)N^j v + \epsilon(t-s+1)N^{j-1} d, j = \min\{s-1, t\} + 1, \dots, l,$$

where $\epsilon(r) = 1$ for $r \geq 0$ and $\epsilon(r) = 0$ for $r < 0$.
Indeed substituting (3) into (2), we get

$$\begin{aligned}
P_i x_k &= \sum_{j=0}^{s-1} C(k, j) \lambda^{k-j} N^j v + \sum_{j=0}^{t-1} \left(\frac{1 - \lambda^k}{(1 - \lambda)^{j+1}} - \sum_{i=0}^{j-1} \frac{C(k, i+1) \lambda^{k-i-1}}{(1 - \lambda)^{j-i}} \right) N^j d \\
&= \sum_{j=0}^{s-1} C(k, j) \lambda^{k-j} N^j v + \sum_{j=0}^{t-1} \frac{1}{(1 - \lambda)^{j+1}} N^j d - \sum_{j=0}^{t-1} \sum_{i=0}^j \frac{C(k, i) \lambda^{k-i}}{(1 - \lambda)^{j-i+1}} N^j d \\
&= \sum_{j=0}^{s-1} C(k, j) \lambda^{k-j} N^j v + B - \sum_{i=0}^{t-1} \sum_{j=i}^{t-1} \frac{C(k, i) \lambda^{k-i}}{(1 - \lambda)^{j-i+1}} N^j d \\
&= \sum_{j=0}^{s-1} C(k, j) \lambda^{k-j} N^j v + B - \sum_{j=0}^{t-1} \sum_{i=j}^{t-1} \frac{C(k, j) \lambda^{k-j}}{(1 - \lambda)^{i-j+1}} N^i d \\
&= \sum_{j=0}^{s-1} C(k, j) \lambda^{k-j} N^j v + B - \sum_{j=0}^{t-1} C(k, j) \lambda^{k-j} \sum_{i=j}^{t-1} \frac{1}{(1 - \lambda)^{i-j+1}} N^i d
\end{aligned}$$

Now (4) follows by considering cases $s > t$, $s = t$ or $s < t$.
Next consider the case $\lambda = 1$. Substituting $S(j, k) = C(k, j+1)$ into (2), we get

$$\begin{aligned}
P_i x_k &= \sum_{j=0}^{s-1} C(k, j) N^j v + \sum_{j=0}^{t-1} C(k, j+1) N^j d \\
&= v + \sum_{j=1}^{s-1} C(k, j) N^j v + \sum_{j=1}^t C(k, j) N^{j-1} d
\end{aligned}$$

Now (5) follows by considering cases $s-1 > t$, $s-1 = t$ or $s-1 < t$.

We now proceed to finish the proof of the theorem:

Case 1: $|\lambda| > 1$. If one of these $A_j, j = 0, \dots, w-1$ is nonzero, then we see from (4) that $\|P_i x_k\| \rightarrow \infty$ as $k \rightarrow \infty$; otherwise $P_i x_k = B$ is a constant vector.

Case 2: $|\lambda| = 1$ and $\lambda \neq 1$. If one of these $A_j, j = 1, \dots, w-1$ is nonzero, then we see from (4) that $\|P_i x_k\| \rightarrow \infty$ as $k \rightarrow \infty$; otherwise the sequence $P_i x_k = B + \lambda^k A_0, k = 1, \dots$, is bounded.

Case 3: $\lambda = 1$. If one of these $B_j, j = 1, \dots, l$ is nonzero, then we see from (5) that $\|P_i x_k\| \rightarrow \infty$ as $k \rightarrow \infty$; otherwise $P_i x_k$ is the constant vector v .

Case 4: $|\lambda| < 1$. We see from (4) that the sequence $P_i x_k, k = 1, \dots$, converges to the constant vector B .

We conclude that for each i , the sequence $P_i x_k, k = 1, \dots$ is either bounded or tends to infinity. If one of these sequences tends to infinity, then since $|x_k| \geq \|P_i x_k\|$, the sequence $x_k, k = 1, \dots$ tends to infinity in norm $|\cdot|$ and

hence in norm $\|\cdot\|$, since the two norms are equivalent. Otherwise all sequences $P_i x_k, k = 1, \dots$ are bounded for all $i = 1, \dots, m$, and from which it follows that $x_k, k = 1, \dots$ is bounded. This completes the proof. Q.E.D.

The following example shows that in infinite dimensional spaces, Theorem 1 is false.

Example 1 For nonnegative integers n , let $c_n = \frac{1}{2}n(n+1)$. Define $\lambda_i = \frac{1}{2}$ for $c_{2n} \leq i \leq c_{2n+1} - 1, n = 0, 1, \dots$ and $\lambda_i = 2$ for $c_{2n-1} \leq i \leq c_{2n} - 1, n = 1, 2, \dots$. Define a linear operator $A : l_2 \rightarrow l_2$ such that $Ae_i = \lambda_i e_{i+1}, i = 0, 1, \dots$. Then the sequence of iterates $A^k e_0 = \lambda_0 \lambda_1 \dots \lambda_{k-1} e_k, k = 1, 2, \dots$ contains subsequences that converge to 0, (e.g. $A^{c_{2n}-1} e_0 = 1/2^n, n = 1, 2, \dots$), subsequences that approach infinity (e.g. $A^{c_{2n}} e_0 = 2^n, n = 1, 2, \dots$), and infinitely many bounded nonconvergent subsequences (e.g. $A^n e_0 = e_n$ for $n = 2, 4, 8, 12, 18, 24, 32, 40, \dots$). Clearly A is bounded but not compact.

Remark 1 For T linear, Theorem 1 appeared in a 1997 unpublished article "On the behavior of the iterates of a matrix" of the author.

In the case that the mapping T in Theorem 1 is linear, we can actually say more:

Theorem 2 Let $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear map. Let $\|\cdot\|$ be a norm on \mathbb{C}^n . Then for any vector $x \in \mathbb{C}^n$, either $\lim_{k \rightarrow \infty} T^k x = 0$ or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$ or $F \leq \|T^k x\| \leq G$ for sufficiently large k 's, where F, G are positive numbers with $F \leq G$.

Proof.

If $c = 0$ in Theorem 1, then the constant vectors B and d in its proof are 0. It follows that in all cases where $P_i x_k, k = 1, 2, \dots$ is bounded and not convergent to 0, it is bounded away from 0. If $P_i x_k$ is bounded away from 0 for some i , then so is $x_k = T^k x$ since $|x_k| \geq \|P_i x_k\|$. Otherwise $P_i x_k \rightarrow 0$ converges to 0 for all i and hence so does x_k . Q.E.D.

Example 2 The following simple example shows that linearity is needed in Theorem 2: Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be the map $Tx = ix + c$, where $c \in \mathbb{C}$ is nonzero and $i = \sqrt{-1}$. Then $T^{4n}(0) = 0$ and $T^{4n+1}(0) = c$ for all positive integers n , showing that $T^k(0), k = 1, 2, \dots$ is neither convergent to 0 nor bounded away from zero.

Definition 1 Let X be a Banach space and $A : X \rightarrow X$ a linear operator. We say A has property (P) if X is a direct sum of two closed subspaces V_1, V_2 such that (1) each V_i is invariant under A , (2) V_1 is finite dimensional, and (3) there

exists $0 \leq r < 1$ and a positive integer N such that $\|A^k x\| \leq r^k \|x\|$ for all $x \in V_2$ and all $k \geq N$.

Note that by Gelfand's spectral radius theorem, condition (3) above is equivalent to that A , as an operator on V_2 , has spectral radius less than 1.

It is well-known that every compact operator, or more generally, Riesz operator, has property (P), see e.g. [1].

Theorem 3 *Let $(X, \|\cdot\|)$ be a Banach space and $A : X \rightarrow X$ a bounded operator having property (P). Let c be a constant vector in X , and let $T(x) = Ax + c$ for $x \in X$. Then for any vector $x \in X$, either $\{T^k x, k = 0, 1, \dots\}$ is bounded or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$. Moreover, if $c = 0$, then either $\lim_{k \rightarrow \infty} T^k x = 0$ or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$ or $F \leq \|T^k x\| \leq G$ for sufficiently large k 's, where $0 < F \leq G < \infty$.*

Proof.

Let $V_i, i = 1, 2, N, r$ be as in Definition 1. Let $P_i, i = 1, 2$ be the projections of X onto $V_i, i = 1, 2$ respectively. Define a norm $|\cdot|$ on X as

$$|x| = \|P_1 x\| + \|P_2 x\|.$$

$|\cdot|$ is equivalent to $\|\cdot\|$ and T commutes with $P_i, i = 1, 2$.

For any $v \in X$, write $v_i = P_i v, i = 1, 2$. Then for $x \in X$, and $k \geq N$ one has

$$\begin{aligned} \|T^k x_2\| &= \|A^k x_2 + c_2 + A c_2 + \dots + A^{k-1} c_2\| \\ &\leq r^k \|x_2\| + \|c_2 + \dots + A^{N-1} c_2\| + (r^N + \dots + r^{k-1}) \|c_2\| \\ &\leq r^k \|x_2\| + \|c_2 + \dots + A^{N-1} c_2\| + \frac{1}{1-r} \|c_2\|, \end{aligned}$$

showing that $T^k x_2, k = 1, 2, \dots$ is bounded. By Theorem 1, $\|T^k x_1\|, k = 1, 2, \dots$ is either bounded or approaching infinity. Hence $|T^k x| = \|T^k x_1\| + \|T^k x_2\|, k = 1, 2, \dots$ is either bounded or approaching infinity. Since norms $|\cdot|$ and $\|\cdot\|$ are equivalent, the same is true for $\|T^k x\|, k = 1, 2, \dots$.

If $c = 0$, then $c_2 = 0$ and $T^k x_2 \rightarrow 0$ as $k \rightarrow \infty$, and the last part of the theorem follows readily from Theorem 2.

Corollary 1 *Let $(X, \|\cdot\|)$ be a Banach space and $A : X \rightarrow X$ a Riesz operator. Let c be a constant vector in X , and let $T(x) = Ax + c$ for $x \in X$. Then for any vector $x \in X$, either $\{T^k x, k = 0, 1, \dots\}$ is bounded or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$. Moreover, if $c = 0$, then either $\lim_{k \rightarrow \infty} T^k x = 0$ or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$ or $F \leq \|T^k x\| \leq G$ for sufficiently large k 's, where $0 < F \leq G < \infty$.*

Corollary 2 *Let $(X, \|\cdot\|)$ be a Banach space and $A : X \rightarrow X$ a compact operator. Let c be a constant vector in X , and let $T(x) = Ax + c$ for $x \in X$. Then for any vector $x \in X$, either $\{T^k x, k = 0, 1, \dots\}$ is bounded or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$. Moreover, if $c = 0$, then either $\lim_{k \rightarrow \infty} T^k x = 0$ or $\lim_{k \rightarrow \infty} \|T^k x\| = \infty$ or $F \leq \|T^k x\| \leq G$ for sufficiently large k 's, where $0 < F \leq G < \infty$.*

Let us consider now the behavior of the sequence of averages of T :

$$\text{Ave}_k T(x) = \frac{1}{k}(x + Tx + \cdots + T^{k-1}x), k = 1, 2, \dots$$

We have

$$\text{Ave}_k T(x) = \frac{1}{k}(x + Ax + \cdots + A^{k-1}x) + \frac{1}{k}((k-1)c + (k-2)Ac + \cdots + A^{k-2}c)$$

As in the proof of Theorem 1, we may assume that $A = \lambda I + N$, and that s, t are defined as in there. By replacing c by x , x by 0, and hence s by 0 and t by s in the proof of Theorem 1, we obtain:

For $\lambda \neq 1$,

$$\frac{1}{k}(x + Ax + \cdots + A^{k-1}x) = \frac{1}{k}B + \sum_{j=0}^{s-1} \lambda^{k-j} \frac{C(k, j)}{k} A_j$$

where

$$A_j = - \sum_{i=j}^{s-1} \frac{1}{(1-\lambda)^{i-j+1}} N^i x, j = 0, \dots, s-1$$

$$B = \sum_{j=0}^{s-1} \frac{1}{(1-\lambda)^{j+1}} N^j x.$$

For $\lambda = 1$,

$$\frac{1}{k}(x + Ax + \cdots + A^{k-1}x) = \frac{1}{k} \sum_{j=1}^s C(k, j) N^{j-1} x.$$

By expanding A^j we have

$$(k-1)c + (k-2)Ac + \cdots + A^{k-2}c = T_0 c + T_1 Nc + \cdots + T_{t-1} N^{t-1} c$$

where

$$T_j = \sum_{i=j}^{k-2} (k-i-1) C(i, j) \lambda^{i-j}, j = 0, \dots, t-1.$$

Assume that $\lambda \neq 1$. By subtracting λT_0 from T_0 and using the geometric series formula, one gets

$$T_0 = \frac{k}{1-\lambda} - \frac{1-\lambda^k}{(1-\lambda)^2}$$

Using the relation $(i-j)C(i, j) = (j+1)C(i, j+1)$, we see that

$$T_{j+1} = \frac{1}{j+1} \frac{d}{d\lambda} T_j$$

We shall prove in the Appendix that

$$T_j = \frac{k}{(1-\lambda)^{j+1}} - \frac{j+1}{(1-\lambda)^{j+2}} + \frac{C(k, j) \lambda^{k-j}}{(1-\lambda)^2} D(k, j, \lambda) \quad (6)$$

for $j = 0, \dots, t-1$, where

$$D(k, j, \lambda) = \frac{1}{(1-\lambda)^j} [B_0(k, j)\lambda^j + B_1(k, j)\lambda^{j-1} + \dots + B_{j-1}(k, j)\lambda + 1]$$

and

$$B_i(k, j) = (-1)^{j-i} C(j, i) \frac{C(k-j, 2)}{C(k-i, 2)}, \quad i = 0, \dots, j$$

Note that

$$B_i(k, j) \rightarrow (-1)^{j-i} C(j, i)$$

so that $D(k, j, \lambda)$ approaches 1 as $k \rightarrow \infty$. Substituting the formulae we obtain thus far into $\text{Ave}_k T(x)$, with $w = \max\{s, t\}$, we get for $\lambda \neq 1$,

$$\text{Ave}_k T(x) = E + \frac{1}{k} F + G(k, \lambda) \quad (7)$$

where

$$G(k, \lambda) = \sum_{j=0}^{w-1} \frac{1}{k} C(k, j) \lambda^{k-j} \left(\epsilon(s-1-j) A_j + \epsilon(t-1-j) \frac{D(k, j, \lambda)}{(1-\lambda)^2} N^j c \right) \quad (8)$$

$$E = \sum_{j=0}^{t-1} \frac{1}{(1-\lambda)^{j+1}} N^j c,$$

$$F = \sum_{j=0}^{s-1} \frac{1}{(1-\lambda)^{j+1}} N^j x - \sum_{j=0}^{t-1} \frac{j+1}{(1-\lambda)^{j+2}} N^j c,$$

and $\epsilon(r) = 1$ for $r \geq 0$ and $\epsilon(r) = 0$ for $r < 0$.

If $\lambda = 1$, then T_j is equal to

$$S_{j,k} = C(k-2, j) + 2C(k-3, j) + \dots + (k-j-1)C(j, j)$$

We have

$$S_{j,k} - S_{j,k-1} = C(k-2, j) + C(k-3, j) + \dots + C(j, j) = C(k-1, j+1)$$

where the last equality follows from repeatedly applying the identity $C(j, i+1) + C(j, i) = C(j+1, i+1)$. From this it follows that

$$\begin{aligned} S_{j,k} &= S_{j,k-1} + C(k-1, j+1) \\ &= S_{j,k-2} + C(k-2, j+1) + C(k-1, j+1) \\ &= \dots \\ &= S_{j,j+2} + C(j+2, j+1) + \dots + C(k-1, j+1) \\ &= C(j+1, j+1) + C(j+2, j+1) + \dots + C(k-1, j+1) \\ &= C(k, j+2) \end{aligned}$$

Therefore for $\lambda = 1$,

$$\begin{aligned} \text{Ave}_k T(x) &= \frac{1}{k} [C(k, 1)x + C(k, 2)Nx + \cdots + C(k, s)N^{s-1}x] \quad (9) \\ &+ \frac{1}{k} [C(k, 2)c + C(k, 3)Nc + \cdots + C(k, t+1)N^{t-1}c] \quad (10) \end{aligned}$$

Case 1: $|\lambda| > 1$. Suppose that $t > s$. Then from (7) and (8), $\frac{1}{k}\lambda^k C(k, t-1)$ dominates all other coefficients. Since $D(k, t-1, \lambda) \rightarrow 1$ as $k \rightarrow \infty$ and $N^{t-1}c \neq 0$, we see that $\|\text{Ave}_k T(x)\| \rightarrow \infty$ as $k \rightarrow \infty$. The same is true if $s > t$ since $A_{s-1} \neq 0$. So assume that $s = t$. In the trivial case $s = t = 0$, we have $\text{Ave}_k T(x) = 0$ for all k . So assume that $s = t \geq 1$. Direct checking (see Appendix, item 1) shows that $C(k, j)C(j, i) \frac{C(k-j, 2)}{C(k-i, 2)} = C(k, i)C(k-i-2, j-i)$ so that $C(k, j)D(k, j, \lambda)$ is a polynomial in k . Then from (8), we have

$$\begin{aligned} G(k, \lambda) &= \sum_{j=0}^{t-1} \frac{1}{k} C(k, j) \lambda^{k-j} \left(A_j + \frac{D(k, j, \lambda)}{(1-\lambda)^2} N^j c \right) \\ &= \frac{\lambda^k}{k} \sum_{j=0}^{t-1} p_1(k, j, \lambda) A_j + p_2(k, j, \lambda) N^j c \end{aligned}$$

where for fixed λ , p_1, p_2 are polynomials in k . Since $\lim_{k \rightarrow \infty} \frac{\lambda^k}{k} p(k) = \infty$ for any polynomial $p(k)$ we see that if

$$H(k, \lambda) = \sum_{j=0}^{t-1} p_1(k, j, \lambda) A_j + p_2(k, j, \lambda) N^j c,$$

as a polynomial in k , is identically zero, then clearly from (7) we have $\text{Ave}_k T(x) \rightarrow E$, otherwise $\|\text{Ave}_k T(x)\| \rightarrow \infty$.

Case 2: $|\lambda| = 1, \lambda \neq 1$. Suppose that $t > s$. Since $D(k, t-1, \lambda) \rightarrow 1$ as $k \rightarrow \infty$ and $N^{t-1}c \neq 0$, we see that $\|\text{Ave}_k T(x)\| \rightarrow \infty$ as $k \rightarrow \infty$ if $t \geq 3$, and it approaches to 0 or is bounded if $t \leq 2$. The same is true if $s > t$ since $A_{s-1} \neq 0$. So assume that $s = t$. In the trivial case $s = t = 0$, we have $\text{Ave}_k T(x) = 0$ for all k . So assume that $s = t \geq 1$. As in Case 1, we consider the polynomial (in k)

$$H(k, \lambda) = \sum_{j=0}^{t-1} C(k, j) \lambda^{-j} \left(A_j + \frac{D(k, j, \lambda)}{(1-\lambda)^2} N^j c \right)$$

so that

$$G(k, \lambda) = \frac{\lambda^k}{k} H(k, \lambda)$$

If degree of H is one or less, then we see from above that $G(k, \lambda)$ is bounded and hence $\text{Ave}_k T(x)$ is bounded by (7). If degree of H is two or more, then $\|G(k, \lambda)\| \rightarrow \infty$ as $k \rightarrow \infty$ and hence so is $\|\text{Ave}_k T(x)\|$.

Case 3: $\lambda = 1$. We refer to (9) and (10). If $s = t = 0$, then $c = x = 0$ and $\text{Ave}_k T(x) = 0$. If $t = 0$ and $s = 1$, then $\text{Ave}_k T(x) = x$. If $t = s - 1, s \geq 2$ and $N^i c + N^{i+1} x = 0$ for all $i = 0, \dots, t - 1$, then $\text{Ave}_k T(x) = x$. In all other cases $\lim_k \|\text{Ave}_k T(x)\| = \infty$.

The previous discussions yield the proof of the following:

Theorem 4 *Let $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an affine map defined by $Tx = Ax + c$, where A is an $n \times n$ complex matrix and c a constant vector in \mathbb{C}^n . Let $\|\cdot\|$ be a norm on \mathbb{C}^n . For any vector $x \in \mathbb{C}^n$, define*

$$\text{Ave}_k T(x) = \frac{1}{k}(x + Tx + \dots + T^{k-1}x)$$

Then the sequence

$$\text{Ave}_k T(x), k = 0, 1, 2, \dots$$

is either bounded or $\lim_k \|\text{Ave}_k T(x)\| = \infty$.

If T is linear, i.e. if $c = 0$, we can say more:

Theorem 5 *Let $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be linear map. Let $\|\cdot\|$ be a norm on \mathbb{C}^n . For any vector $x \in \mathbb{C}^n$, define*

$$\text{Ave}_k A(x) = \frac{1}{k}(x + Ax + \dots + A^{k-1}x)$$

Then the sequence

$$\text{Ave}_k A(x), k = 0, 1, 2, \dots$$

is either (i) convergent to 0, or (ii) $\lim_k \|\text{Ave}_k A(x)\| = \infty$, or (iii) $F \leq \|\text{Ave}_k Ax\| \leq G$ for sufficiently large k 's, where F, G are positive numbers with $F \leq G$.

Proof.

If $c = 0$ in Theorem 4, then $E = 0$, and by examining its proof we see that in all cases where $T^k x$ is bounded and not convergent to 0, it is bounded away from 0. (In Case 2 of the proof, if $H(k, \lambda)$ is of degree one, then $G(k, \lambda)$ is bounded away from 0, and if $H(k, \lambda)$ is of degree 0, i.e. a constant vector, then $G(k, \lambda) \rightarrow 0$.) Q.E.D.

The following example shows that Theorem 5 is false if the map is not linear.

Example 3 Let

$$A = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$$

and let $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Define $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $Tx = Ax + c$. Let $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then

$$\text{Ave}_k T(v) = \frac{1}{k} \frac{1-i^k}{1-i} v + \frac{1-i^{k-1}}{1-i}$$

which does not converge to 0, and has a subsequence converging to 0.

Theorems 4 and 5 are also valid in Banach spaces. For proof, one only has to use these theorems and note that the average of the bounded (resp. null convergent) sequence $T^k x_2, k = 0, 1, \dots$ in the proof of Theorem 3 is also bounded (resp. null convergent). Thus we have

Theorem 6 *Let X be a Banach space. Let $T : X \rightarrow X$ be an affine map defined by $Tx = Ax + c$, where A is an operator with property (P) and c a constant vector in X . Let $\|\cdot\|$ be the norm on X . For any vector $x \in X$, define*

$$\text{Ave}_k T(x) = \frac{1}{k}(x + Tx + \dots + T^{k-1}x)$$

Then the sequence

$$\text{Ave}_k T(x), k = 0, 1, 2, \dots$$

is either bounded or $\lim_k \|\text{Ave}_k T(x)\| = \infty$. Moreover, if $c = 0$, then the sequence

$$\text{Ave}_k A(x), k = 0, 1, 2, \dots$$

is either (i) convergent to 0, or (ii) $\lim_k \|\text{Ave}_k A(x)\| = \infty$, or (iii) $F \leq \|\text{Ave}_k Ax\| \leq G$ for sufficiently large k 's, where $0 < F \leq G < \infty$.

Recall that compact operators, or more generally, Riesz operators have property (P). So the above theorem is valid for these operators.

Our last objective is to prove the following theorem. The result concerns $\text{Ave}_k T(x)$ in the case $s = t$. It shows in particular that H is identically 0 if and only if $x = E$, the unique fixed point of T , i.e. $Tx = x$, so that in case (i) in the proof of Theorem 4 we actually have $\text{Ave}_k T(x) = E$ for all k , not just $\text{Ave}_k T(x) \rightarrow E$ as $k \rightarrow \infty$.

Theorem 7 *Let λ be a fixed complex number, $\lambda \neq 0, 1$. Let*

$$H(k) = \sum_{j=0}^{t-1} C(k, j) \lambda^{-j} \left(A_j + \frac{D(k, j, \lambda)}{(1-\lambda)^2} N^j c \right)$$

be defined as previously. Let $1 \leq i \leq t-1$. The degree of $H(k)$ is at most $i-1$ if and only if

$$N^i x = \frac{1}{(1-\lambda)} N^i c + \frac{1}{(1-\lambda)^2} N^{i+1} c + \dots + \frac{1}{(1-\lambda)^{t-i}} N^{t-1} c \quad (11)$$

$H(k)$ is identically 0 if and only if

$$x = \frac{1}{(1-\lambda)}c + \frac{1}{(1-\lambda)^2}Nc + \cdots + \frac{1}{(1-\lambda)^t}N^{t-1}c \quad (12)$$

The proof will follow from the following discussions.

Consider $D(k, j, \lambda) - 1$. We have

$$D(k, j, \lambda) - 1 = \frac{1}{(1-\lambda)^j} \sum_{i=0}^j (-1)^i C(j, i) \left(\frac{C(k-j, 2)}{C(k-j+i, 2)} - 1 \right) \lambda^i$$

and for $k > j$,

$$\begin{aligned} & \frac{C(k-j, 2)}{C(k-j+i, 2)} - 1 \\ = & i \frac{-2k + 2j - i + 1}{(k-j+i)(k-j+i-1)} \\ = & i \left(\frac{-2}{k} + \frac{2j-i+1}{k^2} \right) \left(1 - \frac{j-i}{k} \right)^{-1} \left(1 - \frac{j-i+1}{k} \right)^{-1} \\ = & \frac{i}{k} \left(-2 + \frac{2j-i+1}{k} \right) \left(1 + \frac{j-i}{k} + \frac{(j-i)^2}{k^2} + \cdots \right) \left(1 + \frac{j-i+1}{k} + \frac{(j-i+1)^2}{k^2} + \cdots \right) \\ = & i \sum_{m=1}^{\infty} \frac{-2(j-i)^{m-1} - (i-1)((j-i)^{m-1} - (j-i+1)^{m-1})}{k^m} \end{aligned}$$

The coefficient of $\frac{1}{k^m}$ in the sum above is obtained from the following calculations:

$$\begin{aligned} & -2 \sum_{p=0}^{m-1} (j-i)^p (j-i+1)^{m-1-p} + (2j-i+1) \sum_{p=0}^{m-2} (j-i)^p (j-i+1)^{m-2-p} \\ = & -2(j-i)^{m-1} - 2(j-i+1) \sum_{p=0}^{m-2} (j-i)^p (j-i+1)^{m-2-p} + (2j-i+1) \sum_{p=0}^{m-2} (j-i)^p (j-i+1)^{m-2-p} \\ = & -2(j-i)^{m-1} + (i-1) \sum_{p=0}^{m-2} (j-i)^p (j-i+1)^{m-2-p} \\ = & -2(j-i)^{m-1} - (i-1)((j-i)^{m-1} - (j-i+1)^{m-1}) \end{aligned}$$

Our first objective is to write $(j-i)^{m-1}$ as a combination of $x_0 = 1, x_1 = i-1, x_2 = (i-2)(i-1), \dots, x_{m-1} = (i-m+1) \cdots (i-1)$. The proof for the following item 1 can be found in [3].

1. Let $f(x) = a_l x^l + \cdots + a_1 x + a_0$ be a polynomial of degree l . Let $x_0 = 1, x_1 = x-1, x_2 = (x-2)(x-1), \dots, x_l = (x-l) \cdots (x-1)$. Then

$$f(x) = d_l x_l + \cdots + d_0 x_0$$

where

$$\begin{aligned}
d_0 &= f(1) \\
d_1 &= f(2) - f(1) \\
d_2 &= \frac{1}{2!} [f(3) - 2f(2) + f(1)] \\
d_3 &= \frac{1}{3!} [f(4) - 3f(3) + 3f(2) - f(1)] \\
&\dots\dots\dots \\
d_l &= \frac{1}{l!} [f(l+1) - C(l,1)f(l) + \dots + (-1)^l f(1)] \\
&= a_l
\end{aligned}$$

Moreover, for any integer n with $n > l$,

$$f(n+1) - C(n,1)f(n) + C(n,2)f(n-1) + \dots + (-1)^n f(1) = 0$$

2. Applying above to the polynomial $f(i) = (j-i)^{m-1}$, we get

$$(j-i)^{m-1} = c_{m-1}x_{m-1} + \dots + c_0x_0$$

where

$$\begin{aligned}
c_0 &= (j-1)^{m-1} \\
c_1 &= (j-2)^{m-1} - (j-1)^{m-1} \\
c_2 &= \frac{1}{2!} [(j-3)^{m-1} - 2(j-2)^{m-1} + (j-1)^{m-1}] \\
c_3 &= \frac{1}{3!} [(j-4)^{m-1} - 3(j-3)^{m-1} + 3(j-2)^{m-1} - (j-1)^{m-1}] \\
&\dots\dots\dots \\
c_{m-1} &= \frac{1}{(m-1)!} [(j-m)^{m-1} - C(m-1,1)(j-m+1)^{m-1} + \dots + (-1)^{m-1}(j-1)^{m-1}] \\
&= (-1)^{m-1}
\end{aligned}$$

Moreover, for any $n > m-1$,

$$(j-n-1)^{m-1} - C(n,1)(j-n)^{m-1} + C(n,2)(j-n+1)^{m-1} + \dots + (-1)^n (j-1)^{m-1} = 0$$

3. Let $x_i, c_i, i = 0, \dots, m-1$ be defined above. Then

$$p(i) = (i-1) [(j-i)^{m-1} - (j-i+1)^{m-1}] = c_1x_1 + 2c_2x_2 + \dots + (m-1)c_{m-1}x_{m-1}$$

so that

$$-2(j-i)^{m-1} - p(i) = -2c_0x_0 - 3c_1x_1 - \dots - (m+1)c_{m-1}x_{m-1}$$

Proof.

Note that the degree of $p(i)$ is $m-1$ and the coefficient of i^{m-1} is

$(-1)^{m-1}(m-1)$. Applying item 1 to $p(i)$ and using the identity $C(k-1, i) + C(k-1, i+1) = C(k, i+1)$ we get

$$\begin{aligned}
d_0 &= p(1) = 0 \\
d_1 &= p(2) - p(1) = (j-2)^{m-1} - (j-1)^{m-1} = c_1 \\
d_2 &= \frac{1}{2!} [p(3) - 2p(2) + p(1)] = (j-3)^{m-1} - (j-2)^{m-1} - [(j-2)^{m-1} - (j-1)^{m-1}] = 2c_2 \\
&\dots\dots\dots \\
d_k &= \frac{1}{k!} [p(k+1) - C(k, 1)p(k) + \dots + (-1)^i C(k, i)p(k-i+1) + \dots + (-1)^{k-1}kp(2) + \\
&\quad (-1)^k p(1)] \\
&= \frac{1}{k!} k [(j-k-1)^{m-1} - C(k, 1)(j-k)^{m-1} + \dots + (-1)^k (j-1)^{m-1}] = kc_k \\
&\dots\dots\dots \\
d_{m-1} &= (-1)^{m-1}(m-1) = (m-1)c_{m-1}
\end{aligned}$$

4. Write $i_0 = i, i_1 = i(i-1), \dots, i_p = i(i-1) \dots (i-p)$. Note that i_p is i times x_p in item 2. This change is necessary because there is an i that was factored out of the summation sign in our expansion of $\frac{C(k-j, 2)}{C(k-j+i, 2)} - 1$. By item 3, the coefficient of $\frac{1}{k^m}$ in the expansion of $\frac{C(k-j, 2)}{C(k-j+i, 2)} - 1$ is $-\sum_{p=0}^{m-1} (p+2)c_p i_p$. Therefore the coefficient $L(m, j, \lambda)$ of $\frac{1}{k^m}$ in the expansion of $D(k, j, \lambda) - 1$ is given by

$$\begin{aligned}
& -\frac{1}{(1-\lambda)^j} \sum_{i=0}^j (-1)^i C(j, i) \left(\sum_{p=0}^{m-1} (p+2)c_p i_p \right) \lambda^i \\
&= -\frac{1}{(1-\lambda)^j} \sum_{p=0}^{m-1} (p+2)c_p \lambda^{p+1} \sum_{i=0}^j (-1)^i C(j, i) i_p \lambda^{i-p-1} \\
&= -\frac{1}{(1-\lambda)^j} \sum_{p=0}^{m-1} (p+2)c_p \lambda^{p+1} \frac{d^{p+1}}{d\lambda^{p+1}} (1-\lambda)^j \\
&= \frac{1}{(1-\lambda)^j} \sum_{p=0}^{m-1} (-1)^p (p+2)c_p \lambda^{p+1} j(j-1) \dots (j-p) (1-\lambda)^{j-p-1} \\
&= \sum_{p=0}^{m-1} (-1)^p (p+2)c_p j(j-1) \dots (j-p) \left(\frac{\lambda}{1-\lambda} \right)^{p+1} \\
&= \sum_{i=1}^m (-1)^{i+1} (i+1)! c_{i-1} C(j, i) \left(\frac{\lambda}{1-\lambda} \right)^i
\end{aligned}$$

To emphasize that c_{i-1} depends on j, m , we write c_i as $P(i, j, m)$, so

$$P(i, j, m) = \frac{1}{i!} [(j-i-1)^{m-1} - C(m-1, 1)(j-i)^{m-1} + \dots + (-1)^{m-1} (j-1)^{m-1}]$$

and

$$L(m, j, \lambda) = \sum_{i=1}^m (-1)^{i+1} (i+1)! P(i-1, j, m) C(j, i) \left(\frac{\lambda}{1-\lambda} \right)^i$$

We have $P(0, j, m) = c_0 = (j-1)^{m-1}$, $P(m-1, j, m) = c_{m-1} = (-1)^{m-1}$, $P(i, j, m) = c_i = 0$ for $i \geq m$. We define $P(i, j, m) = 0$ for negative i . We have the following recursive relations, which will not be used in the sequel,

$$P(i, j, m) = (j-i-1)P(i, j, m-1) - P(i-1, j, m-1), m \geq 2.$$

Also we define

$$M(m, j, x) = \sum_{i=1}^m (-1)^{i+1} (i+1)! P(i-1, j, m) C(j, i) x^i$$

so that

$$L(m, j, \lambda) = M(m, j, \frac{\lambda}{1-\lambda})$$

5.

$$\begin{aligned} & M(m, j, x) - H_1(j_1, \dots, j_{m-1})M(m-1, j, x) + \dots \\ & + (-1)^i H_i(j_1, \dots, j_{m-1})M(m-i, j, x) + \dots + (-1)^{m-1} H_{m-1}(j_1, \dots, j_{m-1})M(1, j, x) \\ & = j_0 j_1 \dots j_{m-1} (m+1) x^m \end{aligned}$$

where $j_i = j - i$, $i = 0, \dots, m-1$ and $H_i(j_1, \dots, j_{m-1})$ is the unsigned coefficient of y^i in the expansion of the polynomial

$$(y - j_1)(y - j_2) \dots (y - j_{m-1})$$

i.e. $H_1 = j_1 + \dots + j_{m-1}$, $H_2 = \sum_{i < k} j_i j_k, \dots, H_{m-1} = j_1 \dots j_{m-1}$.

In particular, we have

$$\begin{aligned} M(1, j, x) &= 2jx \\ M(2, j, x) &= j(j-1)3x^2 + (j-1)M(1, j, x) = j(j-1)(3x^2 + 2x) \\ M(3, j, x) &= j(j-1)(j-2)4x^3 + (j_1 + j_2)M(2, j, x) - j_1 j_2 M(1, j, x) \\ &= j(j-1)((j-2)4x^3 + (2j-3)3x^2 + 2(j-1)x) \end{aligned}$$

Proof.

For fixed m, j ,

$$\begin{aligned} p(x) &= M(m, j, x) - H_1(j_1, \dots, j_{m-1})M(m-1, j, x) + \dots \\ &+ (-1)^i H_i(j_1, \dots, j_{m-1})M(m-i, j, x) + \dots + (-1)^{m-1} H_{m-1}(j_1, \dots, j_{m-1})M(1, j, x) \end{aligned}$$

is a polynomial in x with $p(0) = 0$. Fix $1 \leq k \leq m-1$. We shall show

that the coefficient of x^k in $p(x)$ is 0. Now the coefficient is

$(-1)^{k+1} (k+1)! C(j, k) a_k$, where

$$\begin{aligned} a_k &= P(k-1, j, m) - H_1 P(k-1, j, m-1) + \dots + (-1)^i H_i P(k-1, j, m-i) + \dots \\ &+ (-1)^{m-1} H_{m-1} P(k-1, j, 1) \end{aligned}$$

Since

$$y^{m-1} - H_1 y^{m-2} + \dots + (-1)^i H_i y^{m-i-1} + \dots + (-1)^{m-1} H_{m-1} = (y-j_1) \cdots (y-j_{m-1})$$

we have for each $1 \leq z \leq k$,

$$j_z^{m-1} - H_1 j_z^{m-2} + \dots + (-1)^i H_i j_z^{m-i-1} + \dots + (-1)^{m-1} H_{m-1} = 0$$

From the definition of $P(i, j, m)$, $j_z^{m-1}, j_z^{m-2}, \dots$ are the corresponding terms (with the same coefficient) in $P(k-1, j, m), P(k-1, j, m-1), \dots$. It follows that $a_k = 0$.

For $k = m$, we have $a_k = P(m-1, j, m) = (-1)^{m-1}$ since $P(i, j, m) = 0$, for $i \geq m$. Thus the coefficient of x^m is $(-1)^{m+1}(-1)^{m-1}(m+1)!C(j, m) = (m+1)j_0 j_1 \cdots j_{m-1}$. This completes the proof.

6. Let m_1, m_2, \dots , be variables. Let k, j , be positive integers. Define $R(k, j)$ to be the sum of

$$m_1^a m_2^b \cdots m_k^c$$

where a, b, \dots, c are nonnegative integers such that $0 \leq a, b, \dots, c \leq j$ and $a + b + \dots + c = j$. Define $R(k, 0) = 1$. Define $S(k, j)$ to be the sum of

$$m_1^a m_2^b \cdots m_k^c$$

where a, b, \dots, c are nonnegative integers such that $0 \leq a, b, \dots, c \leq 1$ and $a + b + \dots + c = j$. Define $S(k, 0) = 1$ and $S(0, 0) = 1$. Then we have the following identity:

$$\sum_{i=0}^q (-1)^i R(p+i, q-i) S(p+i-1, i) = 0$$

for any positive integers p, q .

Proof.

Consider a typical term $t = m_1^a m_2^b \cdots m_{p+i}^k$ resulting from the summand $(-1)^i R(p+i, q-i) S(p+i-1, i)$, sign disregarded. Let us write $c(1) = a, c(2) = b, \dots, c(p+i) = k$. We may assume that $c(p+i) \geq 1$, otherwise the term belongs to a previous summand; this assumption also implies that t does not belong to any previous (smaller i) summand. Let

$$A = \{j : 1 \leq j \leq p+i-1, c(j) \neq 0\}$$

Then the cardinality $|A|$ of A must be greater than or equal to i because of $S(p+i-1, i)$. Also the term t appears in the expansion of the summand for exactly $C(|A|, i)$ times. Next let us consider how many times t appears in the expansion of the next summand $(-1)^{i+1} R(p+i+1, q-i-1) S(p+i, i+1)$. t can result from multiplying a term in $R(p+i+1, q-i-1)$ with a term s in $S(p+i, i+1)$. Denote the exponent of m_{p+i} in s by $b_s(p+i)$, which is either 1 or 0.

Denote by S^* and T^* the set of terms in $S(p+i, i+1)$ and $R(p+i+1, q-i-1)$ respectively. Consider the following two sets:

$$U_k = \{s : s \in S^*, b_s(p+i) = k, rs = t \text{ for some } r \in R^*\}, k = 0, 1.$$

Note that each $s \in U_k$ corresponds to exactly one r such that $rs = t$. Clearly U_1 has exactly $C(|A|, i)$ elements, and these elements yields the same number of t 's which cancel out with the previous ones because of the sign change.

Each $s \in U_0$ consists of $i+1$ factors from m_1, \dots, m_{p+i-1} , so U_0 has $C(|A|, i+1)$ elements which yield the same number of t 's.

If $C(|A|, i+1) = 0$, i.e. $|A| < i+1$, S_0 is empty and we are done since no further t 's will result. If not, we consider the next summand $(-1)^i R(p+i+2, q-i-2)S(p+i+1, i+2)$.

Denote by S_1^* and T_1^* the set of terms in $S(p+i+1, i+2)$ and $R(p+i+2, q-i-2)$ respectively. Consider the following two sets:

$$V_k = \{s : s \in S_1^*, b_s(p+i) = k, rs = t \text{ for some } r \in R_1^*\}, k = 0, 1.$$

Note that it is $b_s(p+i)$, not $b_s(p+i+1)$, in the above definition of V_k . Also note that for each $s \in V_k$, $b_s(p+i+1)$ must be 0 or else $rs = t$ is impossible. Then it is clear that $|V_1| = C(|A|, i+1)$ and $|V_0| = C(|A|, i+2)$, as before. The $C(|A|, i+1)$ t 's resulting from V_1 cancel out with the previous the same number of t 's. The same statement about U_0 above applies to V_0 and the process continues. This process will continue for at most $p-1$ times since $|A| \leq i+p-1$. This proves that the t 's occurring in $\sum_{i=0}^q (-1)^i R(p+i, q-i)S(p+i-1, i)$ are all canceled out. Q.E.D.

7. We now finish the proof of Theorem 7. Let $i = t-1$. H has degree of at most $i-1$ if and only if the coefficient of k^{t-1} is 0. Since $\lim_{k \rightarrow \infty} D(k, j, \lambda) = 1$, this amounts to

$$A_{t-1} + \frac{1}{(1-\lambda)^2} N^{t-1} c = 0$$

which is equivalent to

$$N^{t-1} x = \frac{1}{1-\lambda} N^{t-1} c$$

So the assertion in Theorem 7 is true for $i = t-1$. Suppose the assertion is true for some $i \geq 1$; this implies that coefficients of k^j in H are 0 for $j \geq i$. We shall prove that it is also true for $i-1$. This will complete the proof by induction.

By induction hypothesis, we have

$$N^i x = \sum_{j=i}^{t-1} \frac{1}{(1-\lambda)^{j-i+1}} N^j c$$

By applying N to both sides $t - 1 - i$ times, recalling that $N^t c = 0$, we get

$$N^l x = \sum_{j=l}^{t-1} \frac{1}{(1-\lambda)^{j-l+1}} N^j c \quad (13)$$

for all $l = i, \dots, t-1$.

Substituting (13) into the formula for A_l , we find

$$A_l = - \sum_{j=l}^{t-1} \frac{j-l+1}{(1-\lambda)^{j-l+2}} N^j c \quad (14)$$

for all $l = i, \dots, t-1$.

Using $A_{j-1} = \frac{1}{1-\lambda}(A_j - N^{j-1}x)$, we get

$$A_{i-1} = - \sum_{j=i}^{t-1} \frac{j-i+1}{(1-\lambda)^{j-i+3}} N^j c - \frac{1}{1-\lambda} N^{i-1} x \quad (15)$$

If we write P_j for the term

$$C(k, j) \lambda^{-j} \left(A_j + \frac{D(k, j, \lambda)}{(1-\lambda)^2} N^j c \right)$$

in H and substitute (14) and (15), we find that, for $i \leq l \leq t-1$

$$P_l = C(k, l) \lambda^{-l} \left[- \sum_{p=l+1}^{t-1} \frac{p-l+1}{(1-\lambda)^{p-l+2}} N^p c + \frac{D(k, l, \lambda) - 1}{(1-\lambda)^2} N^l c \right] \quad (16)$$

where for $l = t-1$ the sum $\sum_{j=l+1}^{t-1}$ is an empty sum, and hence its value is 0; and

$$P_{i-1} = C(k, i-1) \lambda^{-i+1} \left[- \sum_{j=i}^{t-1} \frac{j-i+1}{(1-\lambda)^{j-i+3}} N^j c - \frac{1}{1-\lambda} N^{i-1} x + \frac{D(k, i-1, \lambda)}{(1-\lambda)^2} N^{i-1} c \right] \quad (17)$$

Since $C(k, j)$ and $C(k, j)D(k, j, \lambda)$ are polynomials in k of degree j , the terms $P_j, j = 0, \dots, i-2$ in H contain only k powers of power less than $i-1$. And since the coefficients of k powers of power greater than $i-1$ are 0 by induction hypothesis, we see that the coefficient of k^{i-1} in H is equal to the limit

$$\lim_{k \rightarrow \infty} \frac{1}{k^{i-1}} (P_{i-1} + P_i + \dots + P_{t-1}) \quad (18)$$

Fix an $m, i \leq m \leq t-1$. Write $A = A(k, m)$ for the number

$$\frac{C(k, m) \lambda^{-m}}{k^m (1-\lambda)^2}$$

The coefficient of the vector $N^m c$ in $\frac{1}{k^{i-1}}P_j$ for each $j, i \leq j \leq m-1$, is, by (16),

$$-C(k, j)\lambda^{-j} \frac{m-j+1}{(1-\lambda)^{m-j+2}}$$

which can be rewritten as

$$A \frac{k^{m-j}}{(k-j) \cdots (k-m+1)} k^{j-i+1} (-(m-j+1)m(m-1) \cdots (j+1) \left(\frac{\lambda}{1-\lambda}\right)^{m-j})$$

and which by item 5 is equal to

$$\begin{aligned} & A \frac{k^{m-j}}{(k-j) \cdots (k-m+1)} k^{j-i+1} (-L(m-j, m, \lambda) + S(m-j-1, 1)L(m-j-1, m, \lambda) + \cdots + \\ & (-1)^{m-j} S(m-j-1, m-j-1)L(1, m, \lambda)) \\ = & A \frac{k^{m-j}}{(k-j) \cdots (k-m+1)} k^{j-i+1} \sum_{l=1}^{m-j} (-1)^{m-j-l+1} S(m-j-1, m-j-l)L(l, m, \lambda) \end{aligned}$$

where S is defined as in item 6, with $m_1 = m-1, \dots$. We also have

$$\begin{aligned} \frac{k^{m-j}}{(k-j) \cdots (k-m+1)} k^{j-i+1} &= k^{j-i+1} (1 + R(m-j, 1)k^{-1} + R(m-j, 2)k^{-2} + \cdots \\ &+ R(m-j, j-i+1)k^{-j+i-1}) + O(1/k) \\ &= \sum_{z=0}^{j-i+1} R(m-j, j-i+1-z)k^z + O(1/k) \end{aligned}$$

where R is defined as in item 6, with $m_1 = m-1, \dots$.

The coefficient of the vector $N^m c$ in $\frac{1}{k^{i-1}}P_m$ is

$$\begin{aligned} & Ak^{m-i+1}(D(k, m, \lambda) - 1) \\ = & Ak^{m-i}(L(1, m, \lambda) + L(2, m, \lambda)k^{-1} + \cdots + L(m-i+1, m, \lambda)k^{-m+i}) + O(1/k) \\ = & A \sum_{z=0}^{m-i} L(m-i+1-z, m, \lambda)k^z + O(1/k) \end{aligned}$$

Note that $N^m c$ does not appear in P_l for $l > m$; so the coefficient of $N^m c$ in the sum

$$\frac{1}{k^{i-1}}(P_i + \cdots + P_{t-1})$$

is the same as that in the sum

$$\frac{1}{k^{i-1}}(P_i + \cdots + P_m).$$

Therefore the coefficient of $N^m c$ in the sum

$$\frac{1}{k^{i-1}}(P_i + \cdots + P_{t-1})$$

which we shall call c_m is A times

$$Y = \sum_{j=i}^{m-1} \sum_{l=1}^{m-j} (-1)^{m-j-l+1} S(m-j-1, m-j-l) L(l, m, \lambda) \sum_{z=0}^{j-i+1} R(m-j, j-i+1-z) k^z + \sum_{z=0}^{m-i} L(m-i+1-z, m, \lambda) k^z + O(1/k)$$

(Note that $A = O(1)$ so $AO(1/k) = O(1/k)$.)

Now we shall show that the polynomial part of Y is a constant, i.e. the coefficients of k^z , $z = 1, \dots, m-i$ are zero.

Fix z , $1 \leq z \leq m-i$. Since k^z occurs in the sum $\sum_{z=0}^{j-i+1} R(m-j, j-i+1-z) k^z$ only when $j-i+1 \geq z$, i.e. $j \geq z+i-1$, we see that the coefficient of k^z in Y is $L(m-i+1-z, m, \lambda)$ plus

$$\begin{aligned} & \sum_{j=z+i-1}^{m-1} \sum_{l=1}^{m-j} (-1)^{m-j-l+1} S(m-j-1, m-j-l) L(l, m, \lambda) R(m-j, j-i+1-z) \\ &= \sum_{l=1}^{m-i-z+1} L(l, m, \lambda) \sum_{j=z+i-1}^{m-l} (-1)^{m-j-l+1} S(m-j-1, m-j-l) R(m-j, j-i+1-z) \\ &= \sum_{l=1}^{m-i-z+1} L(l, m, \lambda) \sum_{n=0}^{m-i-z+1-l} (-1)^{n+1} R(l+n, m-i-z+1-l-n) S(l+n-1, n) \\ &= \sum_{l=1}^{m-i-z} L(l, m, \lambda) \cdot 0 + L(m-i-z+1, m, \lambda) \cdot (-1) \\ &= -L(m-i-z+1, m, \lambda) \end{aligned}$$

where we have used item 5 for the 0. Hence the coefficient of k^z is 0.

The constant term in Y is given by the coefficient of k^0 , which is $L(m-i+1, m, \lambda)$ plus

$$\begin{aligned} & \sum_{j=i}^{m-1} \sum_{l=1}^{m-j} (-1)^{m-j-l+1} S(m-j-1, m-j-l) L(l, m, \lambda) R(m-j, j-i+1) \\ &= \sum_{l=1}^{m-i} L(l, m, \lambda) \sum_{j=i}^{m-l} (-1)^{m-j-l+1} R(m-j, j-i+1) S(m-j-1, m-j-l) \\ &= \sum_{l=1}^{m-i} L(l, m, \lambda) \sum_{n=0}^{m-i-l} (-1)^{n+1} R(l+n, m-l-i+1-n) S(l+n-1, n) \\ &= \sum_{l=1}^{m-i} L(l, m, \lambda) (-1)^{m-i-l-1} R(l+n, 0) S(m-i, m-i-l+1) \\ &= \sum_{l=1}^{m-i} (-1)^{m-i-l-1} S(m-i, m-i-l+1) L(l, m, \lambda) \end{aligned}$$

Adding the term $L(m-i+1, m, \lambda)$ back in we conclude that the said coefficient c_m is

$$A[L(m-i+1, m, \lambda) - S(m-i, 1)L(m-i, m, \lambda) + \dots + (-1)^{m-i}S(m-i, m-i)L(1, m, \lambda)] + O(1/k)$$

which by item 5 is

$$A \frac{m!}{(i-1)!} (m-i+2) \left(\frac{\lambda}{1-\lambda} \right)^{m-i+1} + O(1/k)$$

On the other hand, by (17), the coefficient of $N^m c$ in P_{i-1}/k^{i-1} can be written as

$$-A \frac{m!}{(i-1)!} (m-i+1) \left(\frac{\lambda}{1-\lambda} \right)^{m-i+1} + O(1/k)$$

Therefore the coefficient of $N^m c$, $i \leq m \leq t-1$, in $\frac{1}{k^{i-1}}(P_{i-1} + \dots + P_{t-1})$ is

$$A \frac{m!}{(i-1)!} \left(\frac{\lambda}{1-\lambda} \right)^{m-i+1} + O(1/k),$$

which approaches to

$$\frac{\lambda^{-i+1}}{(i-1)!(1-\lambda)^{m-i+3}}$$

as $k \rightarrow \infty$ since $A = A(k, m) \rightarrow \lambda^{-m}/(m!(1-\lambda)^2)$ as $k \rightarrow \infty$. Since $D(k, i-1, \lambda) \rightarrow 1$, $\frac{C(k, i-1)}{k^{i-1}} \rightarrow \frac{1}{(i-1)!}$, and since N^{i-1} only appears in P_{i-1} , we see that the above sentence is also valid for $m = i-1$. It then follows from (17) and (18) that the coefficient of k^{i-1} in H is

$$\frac{\lambda^{-i+1}}{(i-1)!} \left[\sum_{j=i-1}^{t-1} \frac{1}{(1-\lambda)^{j-i+3}} N^j c - \frac{1}{1-\lambda} N^{i-1} x \right]$$

Therefore H is of degree $i-2$ or less if and only if

$$N^{i-1} x = \sum_{j=i-1}^{t-1} \frac{1}{(1-\lambda)^{j-i+2}} N^j c$$

The last part of the theorem corresponds to $i = 1$ in the above equation. This completes the proof of Theorem 7.

Appendix

We prove by induction that

$$T_j = \frac{k}{(1-\lambda)^{j+1}} - \frac{j+1}{(1-\lambda)^{j+2}} + \frac{C(k, j)\lambda^{k-j}}{(1-\lambda)^2} D(k, j, \lambda)$$

for $j = 0, \dots, t-1$, where

$$D(k, j, \lambda) = \frac{1}{(1-\lambda)^j} [B_0(k, j)\lambda^j + B_1(k, j)\lambda^{j-1} + \dots + B_{j-1}(k, j)\lambda + 1]$$

and

$$B_i(k, j) = (-1)^{j-i} C(j, i) \frac{C(k-j, 2)}{C(k-i, 2)}, \quad i = 0, \dots, j$$

As we noted in the paper proper this is true for $j = 0$ since $D(k, 0, \lambda) = 1$. By item 1 below T_j can be rewritten as

$$\frac{k}{(1-\lambda)^{j+1}} - \frac{j+1}{(1-\lambda)^{j+2}} + \frac{\lambda^{k-j} A}{(1-\lambda)^{j+2}}$$

where

$$A = \sum_{i=0}^j (-1)^{j-i} C(k, i) C(k-i-2, j-i) \lambda^{j-i}$$

Taking the derivative of T_j with respect to λ , we find

$$\frac{d}{d\lambda} T_j = \frac{k(j+1)}{(1-\lambda)^{j+2}} - \frac{(j+1)(j+2)}{(1-\lambda)^{j+3}} + \frac{\lambda^{k-j-1} [(1-\lambda)(k-j) + (j+2)\lambda] A + (1-\lambda)\lambda A'}{(1-\lambda)^{j+3}}$$

Then item 3 below proves that

$$\frac{d}{d\lambda} T_j = \frac{k(j+1)}{(1-\lambda)^{j+2}} - \frac{(j+1)(j+2)}{(1-\lambda)^{j+3}} + (j+1) \frac{C(k, j+1) \lambda^{k-j-1}}{(1-\lambda)^2} D(k, j+1, \lambda)$$

This completes the induction since, as stated in the paper proper,

$$T_{j+1} = \frac{1}{j+1} \frac{d}{d\lambda} T_j$$

1.

$$C(k, j) C(j, i) \frac{C(k-j, 2)}{C(k-i, 2)} = C(k, i) C(k-i-2, j-i)$$

Proof.

$$\begin{aligned} & C(k, j) C(j, i) \frac{C(k-j, 2)}{C(k-i, 2)} \\ &= \frac{k! j! (k-j)! (k-i-2)! 2!}{j! (k-j)! (j-i)! i! (k-j-2)! 2! (k-i)!} \\ &= \frac{k! (k-i-2)!}{i! (k-i)! (k-j-2)! (j-i)!} \\ &= C(k, i) C(k-i-2, j-i) \end{aligned}$$

Q.E.D.

2. For $k \geq i + 2, j \geq i - 1$,

$$\begin{aligned} & (k - i + 1)C(k, i - 1)C(k - i - 1, j - i + 1) + (k - i - j - 2)C(k, i)C(k - i - 2, j - i) \\ = & (j + 1)C(k, i)C(k - i - 2, j - i + 1). \end{aligned}$$

Proof.

$$\begin{aligned} & (k - i + 1)C(k, i - 1)C(k - i - 1, j - i + 1) + (k - i - j - 2)C(k, i)C(k - i - 2, j - i) \\ = & (k - i + 1) \frac{k!(k - i - 1)!}{(k - i + 1)!(i - 1)!(k - j - 2)!(j - i + 1)!} \\ & + (k - i - j - 2) \frac{k!(k - i - 2)!}{(k - i)!i!(k - j - 2)!(j - i)!} \\ = & \frac{ik!(k - i - 1)!}{(k - i)!i!(k - j - 2)!(j - i + 1)!} \\ & + (k - i - j - 2) \frac{(j - i + 1)k!(k - i - 2)!}{(k - i)!i!(k - j - 2)!(j - i + 1)!} \\ = & \frac{k!(k - i - 2)![i(k - i - 1) + (k - i - j - 2)(j - i + 1)]}{(k - i)!i!(k - j - 2)!(j - i + 1)!} \\ = & \frac{k!(k - i - 2)![kj + k - j^2 - 3j - 2]}{(k - i)!i!(k - j - 2)!(j - i + 1)!} \\ = & \frac{k!(k - i - 2)!(j + 1)(k - j - 2)}{(k - i)!i!(k - j - 2)!(j - i + 1)!} \\ = & \frac{k!(k - i - 2)!(j + 1)}{(k - i)!i!(k - j - 3)!(j - i + 1)!} \\ = & (j + 1)C(k, i)C(k - i - 2, j - i + 1) \end{aligned}$$

Q.E.D.

3. Let

$$A = \sum_{i=0}^j (-1)^{j-i} C(k, i)C(k - i - 2, j - i) \lambda^{j-i}$$

and let $A' = \frac{d}{d\lambda} A$. Then

$$\begin{aligned} & [(1 - \lambda)(k - j) + (j + 2)\lambda]A + (1 - \lambda)\lambda A' \tag{19} \\ = & (j + 1) \sum_{i=0}^{j+1} (-1)^{j+1-i} C(k, i)C(k - i - 2, j + 1 - i) \lambda^{j+1-i} \tag{20} \end{aligned}$$

Proof. We have

$$A' = \sum_{i=0}^{j-1} (-1)^{j-i} (j - i) C(k, i)C(k - i - 2, j - i) \lambda^{j-i-1}$$

It is easy to see that

$$\begin{aligned} & (1 - \lambda)[(k - j)A + \lambda A'] \\ = & (1 - \lambda) \sum_{i=0}^j (-1)^{j-i} (k - i) C(k, i) C(k - i - 2, j - i) \lambda^{j-i} \end{aligned}$$

and

$$\begin{aligned} & -\lambda \sum_{i=0}^j (-1)^{j-i} (k - i) C(k, i) C(k - i - 2, j - i) \lambda^{j-i} + (j + 2) \lambda A \\ = & \sum_{i=0}^j (-1)^{j-i+1} (k - i - j - 2) C(k, i) C(k - i - 2, j - i) \lambda^{j-i+1}. \end{aligned}$$

Thus (19) is the sum of

$$B = \sum_{i=0}^j (-1)^{j-i} (k - i) C(k, i) C(k - i - 2, j - i) \lambda^{j-i}$$

and

$$C = \sum_{i=0}^j (-1)^{j-i+1} (k - i - j - 2) C(k, i) C(k - i - 2, j - i) \lambda^{j-i+1}.$$

The coefficient of λ^{j+1-i} in $B + C$ is

$$(-1)^{j-i+1} [(k-i+1)C(k, i-1)C(k-i-1, j-i+1) + (k-i-j-2)C(k, i)C(k-i-2, j-i)],$$

valid even when $i = 0$ or $j + 1$ since by convention $C(n, x) = 0$ for $x < 0$.

This is equal to the coefficient of λ^{j+1-i} in (20) by item 2 above. Q.E.D.

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